Numerically Determining the Dispersion Relations of Nonlinear TE Slab-Guided Waves in Non-Kerr-Like Media

Jian-Guo Ma, *Senior Member, IEEE,* and Zhizhang Chen, *Senior Member, IEEE*

***Abstract*—Dispersion relations are calculated numerically for nonlinear slab guides with a nonlinear core and linear claddings by using the transfer matrix method (TMM). The core can have any nonlinearities while high computation efficiency can still be achieved. As a result, a simple numerical calculation tool without spurious roots is presented for general nonlinear slab-guided structures.**

***Index Terms*—Dispersion, non-Kerr-like nonlinearity, transfer matrix method.**

I. Introduction

CONSIDERABLE interest is currently being directed to nonlinear propagation in nonlinear slab waveguides for integrated optics applications because of the new propagation properties [1]–[17] and their potential use in nonlinear optical devices such as optical switches and nonlinear directional couplers [8].

Nonlinear slab waveguides can be distinguished by the location of the nonlinear layer(s). Typically, waveguides can be etched in two ways: a linear core cladded by one or two nonlinear media [10]–[12], or, the other way around, with a nonlinear core surrounded by two linear claddings [1], [3]–[9], [13]–[17].

Most of the theoretical investigations of these two kinds of nonlinear waveguides have been limited to Kerr-like nonlinearity [3]–[9], [17]. However, "in real media, it is not possible to optically change the refractive index independently and saturation effects eventually set in ... many materials exhibit a refractive index which varies with the optical field raised to a power other than two" [10]. For general non-Kerr-like nonlinearities, most of the studies have been focused on waveguides made of a thin linear film interfaced on either one side or both sides with nonlinear materials [10]–[12]. For the case of a non-Kerr-like nonlinear core with linear claddings, Langbein *et al.* [13] and AL-Bader *et al.* [14] have used a method based on the first integral of the nonlinear Helmholtz equation to deal with the effects of both nonquadratic power-law dependence and saturation of the refractive index on nonlinear guided waves. The solution of the eigenvalue equation can only be obtained after removing a characteristic singularity in the method [14] and must be used cautiously to avoid the confusing spurious roots present in [7]. Rozzi *et al.* [1] used the phase-plane method to study the wave propagation in nonquadratic power-law dependence film. This method is rigorous and can provide a physical interpretation of the results by means of *integrals of motion* in the phase plane. Again, this method is based on the integral of the nonlinear Helmholtz equation.

In [8], a transfer matrix method (TMM) has been developed to study the first kind of nonlinear waveguides with Kerr-like nonlinearity. The structure studied by Ramadas *et al.* [8] is a linear core clad by another linear medium and supported by semi-infinite Kerr-like nonlinear medium, which can be studied analytically [11]. By using the TMM, the values of the field from one boundary are transmitted to another by involving multiplication of 2×2 transfer matrixes only. In [17], Ma has used this method to discuss nonlinear propagations in a Kerr-like nonlinear core surrounded by linear media.

In this paper, the TMM is modified and extended to solve a more general nonlinear case: the nonlinear propagation in nonlinear slab guides with linear claddings and the nonlinear core with a refractive index of nonquadratic power-law dependence. Unlike the method used in [8], global coordinates, instead of local coordinates, are used in this paper's approach to circumvent difficulties when the local coordinates are employed to treat a thin subregion.

The order of the solved matrixes employed in the TMM is still two and an iterative process is not required. In consequence, large central processing unit (CPU) time is not needed and spurious solutions are not found. For simplicity, only TE-waves are studied in this paper.

II. Numerical Method

The schematic drawing of a three-layered slab guide with a non-Kerr-like nonlinear guiding film bounded by linear media is shown in Fig. 1. The film is assumed to have a relative dielectric constant ϵ_2 made of a linear part ϵ_{r2} and an intensity-induced nonlinear part $f(|\vec{E}|^2)$:

$$\epsilon_2 = \epsilon_{r2} + f(|\vec{E}|^2). \tag{1}$$

Here, f represents any nonlinear function. The claddings are assumed to be linear with relative dielectric constant ϵ_{r1} in the substrate $(x < 0)$ and ϵ_{r3} in the cladding $(x > d)$.

Restricting oneself to TE waves and considering only the y-component of the electric field being nonzero, one sees that

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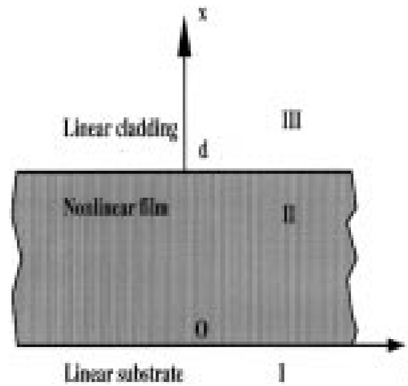

Fig. 1. A slab guide with a nonlinear core, a linear cladding, and a substrate.

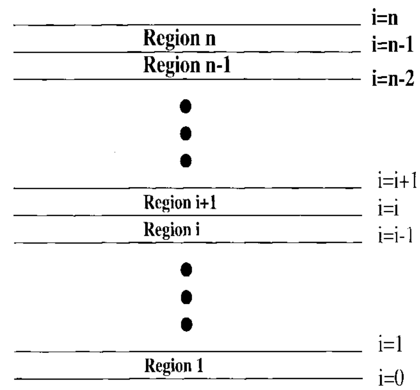

Fig. 2. Dividing the nonlinear Region II into n subregions.

$\vec{E} = \{0, E_y, 0\}$ and $\vec{H} = \{H_x, 0, H_z\}$. The electric field E_y, propagating along the z-axis, corresponds to free space, and must then satisfy the following Helmholtz wave equation [14] in Region II:

$$\frac{d^2E_y}{dx^2} + k_o^2(\epsilon_{r2} - N^2 + f(|E_y|^2)E_y = 0, \qquad 0 \geq x \geq d \quad (2)$$

and in Regions I and III:

$$\frac{d^2E_y}{dx^2} - k_o^2(N^2 - \epsilon_{r1})E_y = 0, \qquad x \leq 0 \quad (3)$$

$$\frac{d^2E_y}{dx^2} - k_o^2(N^2 - \epsilon_{r3})E_y = 0, \qquad x \geq d. \quad (4)$$

Here, E_y is assumed to be in the form of $e^{-j(k_oNz-\omega t)}$ where N is the mode index and k_o corresponds to the free-space wavenumber.

Let E_o be the magnitude of the electric field at $x = 0$, $E_o = E_y(0)$, and take the normalization form of $E_y(x)$ to E_o, $E(x) = E_y(x)/E_o$. It can then be found that the solutions in the linear regions are [1], [3]–[17] as follows:

$$E(x) = \begin{cases} e^{k_1 x}, & x \leq 0, \quad k_l^2 = k_o^2(N^2 - \epsilon_{r1}) \\ \bar{E}_o e^{-\bar{k}_l(x-d)}, & x \geq d, \quad \bar{k}_l^2 = k_o^2(N^2 - \epsilon_{r3}) \end{cases}. \quad (5)$$

To obtain the solutions, Region II (the nonlinear region) is divided into n subregions (as shown in Fig. 2) and $0 = x_o < x_1 < x_2 < \cdots < x_{i-1} < x_i < x_{i+1} < \cdots < x_{n-1} < x_n = d$. In each subregion $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \cdots, n)$, one can assume that the variation of E is small. As a result, the field value at $x = x_{i-1}$, $E(x_{i-1})$ can be used to replace $E(x)$ in the term $f(|E(x)|^2)$ within the whole subregion $x \in [x_{i-1}, x_i]$. Equation (2) is then linearized and becomes a linear equation in each subregion $x \in [x_{i-1}, x_i]$. Its solution is similar to that in a linear dielectric slab guide. Let $E_i(x) = E(x)$, $x \in [x_{i-1}, x_i]$ $(i = 1, 2, \cdots, n)$. Then

$$E_i(x) = A_i \sin(k_i x) + B_i \cos(k_i x) \quad (6)$$

and

$$\frac{dE_i(x)}{dx} = k_i(A_i \cos(k_i x) - B_i \sin(k_i x)) \quad (7)$$

with

$$k_i^2 = k_o^2(\epsilon_{r2} - N^2 + f(|E_i(x_{i-1})|^2)), \qquad (i = 1, 2, \cdots, n) \quad (8)$$

and A_i, B_i being the constants to be determined.

By applying the boundary condition between every two neighboring subregions, the unknown constants A_i, B_i for different subregions can be connected. From the Maxwell's equations, it can be found that $H_{zi}(x) \propto \frac{dE_i(x)}{dx}$. By using the field continuity conditions at interfaces $x = x_{i-1}$ and $x = x_i$, A_i, B_i, and field values of the neighboring subregions are connected as follows:

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = [M_i(x_{i-1})]^{-1}[M_{i-1}(x_{i-1})] \begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix} \quad (9)$$

$$\begin{bmatrix} E_i(x_i) \\ \frac{dE_i(x_i)}{dx} \end{bmatrix} = [M_i(x_i)][M_i(x_{i-1})]^{-1} \begin{bmatrix} E_{i-1}(x_{i-1}) \\ \frac{dE_{i-1}(x_{i-1})}{dx} \end{bmatrix} \quad (10)$$

with

$$M_i(x_{i-1}) = \begin{bmatrix} \sin(k_i x_{i-1}) & \cos(k_i x_{i-1}) \\ k_i \cos(k_i x_{i-1}) & -k_i \sin(k_i x_{i-1}) \end{bmatrix}, \quad i = 1, 2, \cdots, n. \quad (11)$$

Repeated applications of (9) and (10) throughout all the subregions lead to the connections of the coefficients in the first region A_1 and B_1 to the coefficients A_n and B_n in the last subregion n. That is,

$$\begin{bmatrix} E_n(x_n) \\ \frac{dE_n(x_n)}{dx} \end{bmatrix} = \left\{ \prod_{i=2}^{n} [M_i(x_i)][M_i(x_{i-1})]^{-1} \right\} \begin{bmatrix} E_2(x_1) \\ \frac{dE_2(x_1)}{dx} \end{bmatrix} = \left\{ \prod_{i=2}^{n} [M_i(x_i)][M_i(x_{i-1})]^{-1} \right\} [M_1(x_1)] \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}. \quad (12)$$

Using the boundary conditions at $x = x_o = 0$ and $x = x_n = d$, one finds

$$\bar{E}_o = E_n(x_n)$$
$$-\bar{k}_c \bar{E}_o = \frac{dE_n(x_n)}{dx} \quad (13)$$

$$A_1 = \frac{k_l}{k_1}$$
$$B_1 = 1. \quad (14)$$

By substituting (13) and (14) into (12), the following dispersion equation is obtained:

$$\bar{E}_o \begin{bmatrix} 1 \\ -\bar{k}_l \end{bmatrix} = \left\{ \prod_{i=2}^{n} [M_i(x_i)][M_i(x_{i-1})]^{-1} \right\} \begin{bmatrix} \frac{k_l}{k_1} \\ 1 \end{bmatrix}. \quad (15)$$

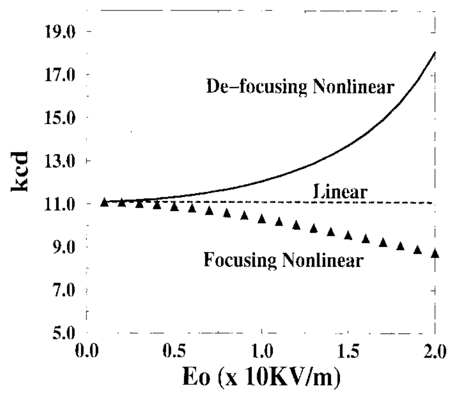

Fig. 3. The cutoff wavenumber of TE_1 versus E_o for Kerr-like nonlinearity ($\alpha = \pm 1.62510^{-10}$ (V/m)^2, $\epsilon_{r1} = \epsilon_{r3} = 3.42,\ \epsilon_{r2} = 3.5$).

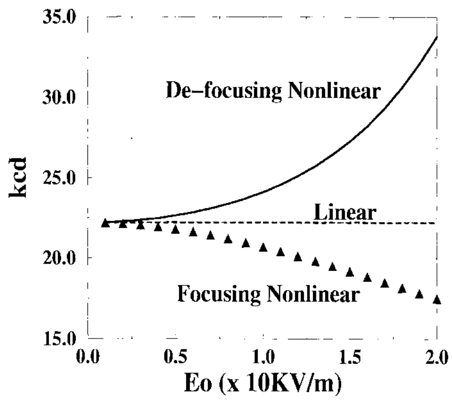

Fig. 4. The cutoff wavenumber of TE_2 versus E_o for Kerr-like nonlinearity ($\alpha = \pm 1.62510^{-10}$ (V/m)^2, $\epsilon_{r1} = \epsilon_{r3} = 3.42,\ \epsilon_{r2} = 3.5$).

Thus, the boundary values at $x = x_o = 0$ are transmitted to the boundary values at $x = d$ (the other side of the nonlinear region).

The dispersion equation (15) is general and it can be used for any nonlinearity. The accuracy can be simply improved by increasing the number n of the subregions.

It is worth mentioning that in [16], the authors have also used the idea of a transmitting matrix to study the nonlinear guided waves in multilayer systems. In each nonlinear layer with Kerr-like nonlinearity, they used the method of the first integral of the wave equation, and the solution in the layer is the Jacobian elliptic function, which is limited to only the Kerr-like cases. Here, the nonlinear equation is linearized at first, and then the well-solved solutions are used for a linear film as the trial solution for the linearized subregions to obtain the final nonlinear solutions. The technique is good for any nonlinearity besides Kerr-nonlinearity.

III. Results

A symmetrical nonlinear dielectric slab guide is computed here with the parameters $\epsilon_{r1} = \epsilon_{r3} = 3.42$, $\epsilon_{r2} = 3.5$, $d = 1\ \mu$m, and

$$f(|E|^2) = \alpha |E_y|^\delta = \gamma \left| \frac{E_y}{E_o} \right|^\delta = \gamma |E|^\delta. \qquad (16)$$

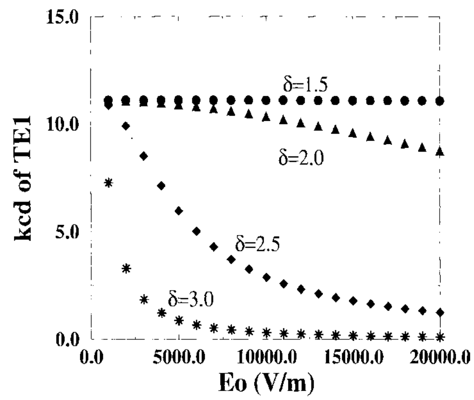

Fig. 5. The cutoff wavenumber of TE_1 versus E_o for focusing non-Kerr-like nonlinearity. Parameters are the same as those in Fig. 4 and $\alpha = 1.62510^{-10}$ (V/m)$^\delta$.

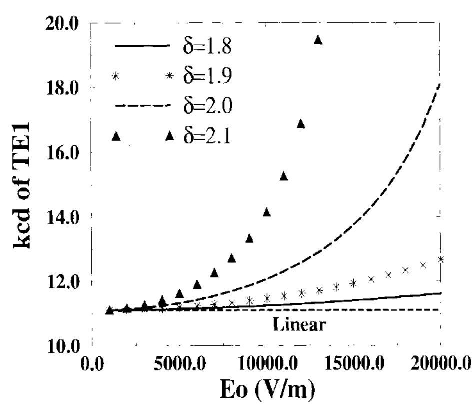

Fig. 6. The cutoff wavenumber of TE_1 versus E_o for de-focusing non-Kerr-like nonlinearity. Parameters are the same as those in Fig. 4 and $\alpha = -1.62510^{-10}$ (V/m)$^\delta$.

$\gamma > 0$ is for focusing nonlinearity and $\gamma < 0$ is for de-focusing nonlinearity.

For the first mode, TE_0, its cutoff wavenumber is zero and is independent of E_o. However, for the other modes, the cutoff wavenumber k_c is dependent on E_o (this is different from linear cases where k_c is always independent of E_o). As examples, nonlinear TE_1 and TE_2 modes are computed, and the k_c dependence on E_o is shown in Figs. 3 and 4 for both focusing and de-focusing Kerr-like nonlinearities. For non-Kerr-like nonlinearities, the k_c dependence is shown in Figs. 5 and 6 with $\alpha = \pm 1.625 \times 10^{-10}$ (V/m)$^\delta$. It can be seen that for focusing nonlinearities, the cutoff wavenumber $k_c d$ is smaller than that of linear cases. However, for a de-focusing nonlinear core, the $k_c d$ is larger than that of linear cases.

In Figs. 7 and 8, the dispersion curves for various δ are shown. It can be seen from the figures that the dispersion curves for a focusing nonlinear core are always above the linear dispersion curves, while for a de-focusing slab guide they are below the curves.

From the above results, it is also found that the mode index N is dependent on the nonlinear coefficient α and the magnitude of the field at $x = 0$, E_o. In other words, N depends on $\gamma = \alpha E_o^\delta$. This dependence is illustrated in Fig. 9 at $k_o d = 4.0$ for TE_0 modes. To ensure propagating modes, N

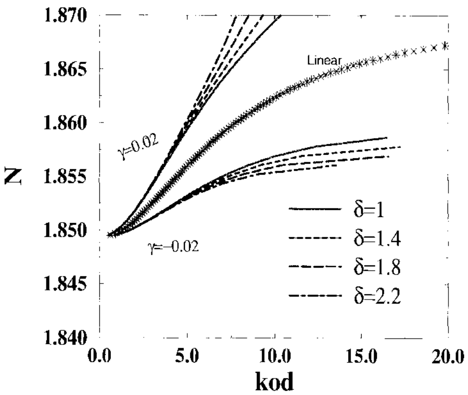

Fig. 7. The dispersion curves of TE_0 mode for various δ.

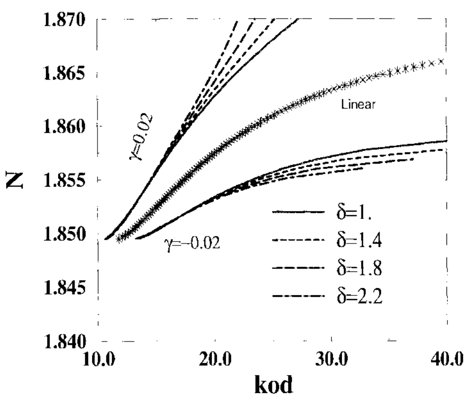

Fig. 8. The dispersion curves of TE_1 mode for various δ.

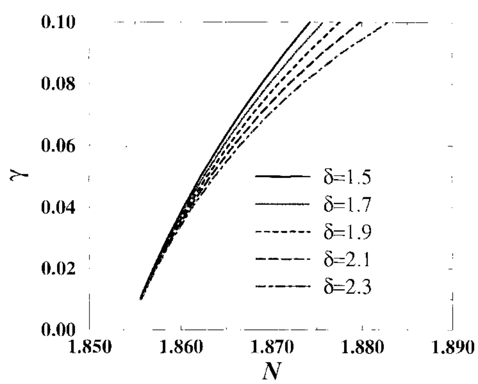

Fig. 9. γ versus N for various δ at $k_o d = 4$ of TE_0 modes.

must satisfy $\sqrt{\epsilon_{r1}} < N < \sqrt{\epsilon_{r2}}$ for linear cases. For nonlinear cases, N can be larger than $\sqrt{\epsilon_{r2}}$ as shown in Fig. 9 with $\epsilon_{r1} = \epsilon_{r3} = 3.42$, $\epsilon_{r2} = 3.5$.

To study the impact of n (number of the linearized subregions) on the accuracy of the solutions, the relative convergence error is calculated. The relative convergence error at n is defined as

$$\frac{k_c d|_{n\,\text{sub-layers}} - k_c d|_{(n-1)\,\text{sub-layers}}}{k_c d|_{n\,\text{sub-layers}}}. \tag{17}$$

In Figs. 10 and 11, the relative convergence errors of the cutoff wavenumber $k_c d$ for TE_1 and TE_2 are shown. The toward-zero tendency of the convergence error gives us the confidence on the validity of the technique used here.

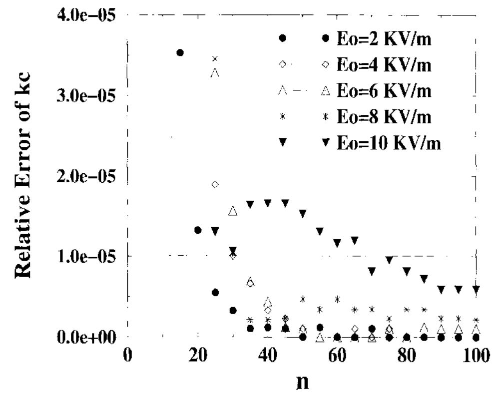

Fig. 10. The relative convergence errors of the cutoff wavenumber $k_c d$ for TE_1 mode. The parameters are the same as those used in the above figures.

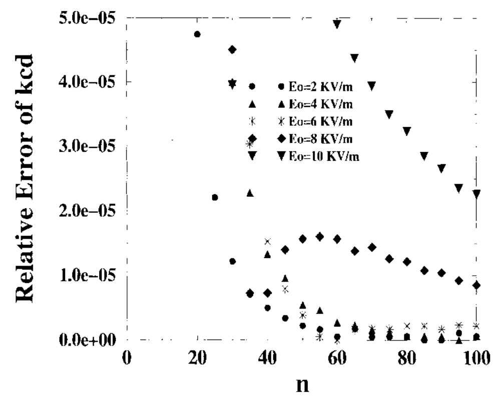

Fig. 11. The relative convergence errors of the cutoff wavenumber $k_c d$ for TE_2 mode. The parameters are the same as those used in the above figures.

IV. Conclusion

In this paper, the TMM is extended to find the dispersion relations of a general nonlinear dielectric slab guide. A number of examples are calculated for the design of optical devices based on nonlinear waveguide structures. The numerical results show that the method has no spurious solutions and yields accurate results when the number of the subregions is sufficient. This makes the method an efficient tool for design purposes.

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